Non-negative Wigner functions for orbital angular momentum states

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The Wigner function of a pure continuous-variable quantum state is non-negative if and only if the state is Gaussian. Here we show that for the canonical pair angle and angular momentum, the only pure states with non-negative Wigner functions are the eigenstates of the angular momentum. Some implications of this surprising result are discussed.

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For continuous variables, the Wigner function [1] is a very useful tool that establishes a one-to-one correspondence between quantum states and joint quasiprobability distributions of canonically conjugate variables in phase space (position and momentum, in the standard case). However, it can take on negative values, a property that distinguishes it from a true probability distribution [2, 3, 4]. Indeed, this negative character is associated with the existence of quantum interference, which itself may be identified as a signal of nonclassical behavior [5].

In consequence, the characterization of quantum states that are classical, in the sense of giving rise to non-negative Wigner functions, is a topic of undoubted interest. Among pure states, it was proven in a classical paper by Hudson [6] (later generalized by Soto and Claverie [7] to multipartite systems) that the only states that have non-negative Wigner functions are Gaussian states [8, 9]. This is one of the main reasons for the prominent role these states play in modern quantum information [10].

The original definition of the Wigner function has also been extended to discrete systems (see Ref. [11] for a comprehensive review). Again, the classification of states with non-negative Wigner functions is an amazing problem that has been solved quite recently by Paz and coworkers [12, 13] and Gross [14, 15], so that the role of Gaussian states is now taken on by stabilizer states. Interestingly, these are the only states that can be simulated efficiently in classical computers [16].

Between these two cases (whose proofs are otherwise completely different), we have the interesting situation of canonical pairs, such as the angle and orbital angular momentum (OAM), for which one variable is continuous while the other one is discrete [17]. The associated phase space is the discrete cylinder $\mathcal{S}_1 \times \mathbb{Z}$, where $\mathcal{S}_1$ stands for the unit circle (associated to the angle) and the integers $\mathbb{Z}$ translate the discreteness of the OAM. The physical example we have in mind is the OAM of photons. This is an emerging field that has given rise to many developments, ranging from optical tweezers to high-dimensional quantum entanglement, or fundamental processes in Bose-Einstein condensates, to cite only a few relevant examples [18].

The seminal paper of Allen *et al.* [19] firmly established that the Laguerre-Gauss modes carry a well-defined OAM. They appear as annular rings with a zero on-axis intensity and an azimuthal dependence $\exp(i\ell\phi)$ that gives rise to spiral wave fronts. The index ℓ takes only integer values and can be seen as the eigenvalue of the OAM operator. Since then, several methods have been established to produce light beams with the required azimuthal phase structure, among these spiral phase plates, forked holograms, and spatial light modulators are perhaps the most versatile. In this way, a variety of modes with helical phase fronts but different transverse patterns (such as Bessel, Mathieu, or hypergeometric beams) can be routinely generated in the laboratory [20].

The goal of this work is precisely to determine the pure states of these OAM-carrying systems for which the Wigner function is non-negative, filling in this way a long overdue gap.

To be as self-contained as possible, we first introduce some basic notions for the problem at hand of cylindrical symmetry. We are concerned with the planar rotations by an angle ϕ generated by the angular momentum along the z axis, which for simplicity will be denoted henceforth as $\hat{L}$. We do not want to enter in a long discussion about the possible existence of an angle operator [21]. For our purposes here, the simplest solution is to adopt two periodic angular coordinates, e.g., cosine and sine, that we shall denote by $\hat{C}$ and $\hat{S}$ to make no further assumptions about the angle itself. One can concisely condense all this information using the complex exponential of the angle $\hat{E} = \hat{C} + i\hat{S}$, which satisfies the commutation relation

$$[\hat{E}, \hat{L}] = \hat{E} \,. \tag{1}$$

In mathematical terms, this defines the Lie algebra of the two-dimensional Euclidean group E(2), which is precisely the canonical symmetry group for the cylinder.

The action of $\hat{E}$ on the basis of eigenstates of $\hat{L}$ is $\hat{E}|\ell\rangle = |\ell - 1\rangle$, and it possesses then a simple implementation by means of a phase mask removing a charge $+1$ from a vortex state [22, 23]. Since the integer ℓ runs from $-\infty$ to $+\infty$, $\hat{E}$ is a unitary operator whose eigenvectors

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{\ell \in \mathbb{Z}} e^{i\ell\phi} |\ell\rangle \tag{2}$$

form a complete basis and describe states with well-defined angle. In the representation generated by them, $\hat{L}$ acts as $-i\partial_\phi$ (in units of $\hbar = 1$).

Given the key role played by the displacement operators in settling the Wigner function for the harmonic oscillator, we introduce a unitary displacement operator

$$\hat{D}(\ell,\phi) = e^{i\alpha(\ell,\phi)}\,\hat{E}^{-\ell} e^{-i\phi\hat{L}}\,, \tag{3}$$

where $\alpha(\ell,\phi)$ is a phase required to avoid plugging in extra factors when acting with $\hat{D}$. The conditions of unitarity and periodicity restrict the possible values of α, although a sensible choice is $\alpha(\ell,\phi) = -\ell\phi/2$. Note that here we cannot rewrite Eq. (3) as an entangled exponential, since the action of the operator to be exponentiated would not be well defined.

We use as a guide the analogy with the continuous case and introduce the mapping [24]

$$W_{\hat{\varrho}}(\ell,\phi) = \mathrm{Tr}[\hat{\varrho}\,\hat{w}(\ell,\phi)]\,, \tag{4}$$

which maps the density operator into a Wigner function via a kernel $\hat{w}$ defined as a double Fourier transform of the displacement operator [25]:

$$\hat{w}(\ell,\phi) = \frac{1}{(2\pi)^2}\sum_{\ell'\in\mathbb{Z}}\int_{2\pi}\exp[-i(\ell'\phi-\ell\phi')]\,\hat{D}(\ell',\phi')\,d\phi'\,, \tag{5}$$

where the integral extends to the 2π interval within which the angle is defined. This mapping is invertible, so one can reconstruct the density operator as

$$\hat{\varrho} = 2\pi\sum_{\ell\in\mathbb{Z}}\int_{2\pi}\hat{w}(\ell,\phi)\,W_{\hat{\varrho}}(\ell,\phi)\,d\phi\,. \tag{6}$$

The (Hermitian) Wigner kernels $\hat{w}(\ell,\phi)$ are a complete orthonormal basis (in the trace sense) for the operators acting on the Hilbert space of the system. In addition, they are explicitly covariant; i.e., they transform properly under displacements, $\hat{w}(\ell,\phi) = \hat{D}(\ell,\phi)\,\hat{w}(0,0)\,\hat{D}^{\dagger}(\ell,\phi)$. In fact, these properties guarantee that the Wigner function defined in Eq.(4) bears all the good properties required for a probabilistic description. In particular, it reproduces the proper marginal distributions, that is,

$$\sum_{\ell\in\mathbb{Z}} W_{\hat{\varrho}}(\ell,\phi) = \langle\phi|\hat{\varrho}|\phi\rangle\,, \qquad \int_{2\pi} W_{\hat{\varrho}}(\ell,\phi)\,d\phi = \langle\ell|\hat{\varrho}|\ell\rangle\,. \tag{7}$$

Finally, the overlap of two density operators is proportional to the integral of the associated Wigner functions:

$$\mathrm{Tr}(\hat{\varrho}\,\hat{\sigma}) \propto \sum_{\ell\in\mathbb{Z}}\int_{2\pi} W_{\hat{\varrho}}(\ell,\phi) W_{\hat{\sigma}}(\ell,\phi)\,d\phi\,. \tag{8}$$

This property (often called traciality) offers practical advantages, since it allows one to predict the statistics of any outcome, once the Wigner function of the measured state is known.

We remark that this approach to the Wigner function is grounded in the axiomatic method developed by Stratonovich [26] and Berezin [24] (see also Ref. [27]). It is possible to follow alternative routes, such as, introducing a Wigner function as the Fourier transform of some generalized characteristic function [28]. This has been pursued also for the group E(2) [29]. However, these apparently disjoint formulations turn out to be equivalent for most practical purposes [30].

To give an explicit form of the Wigner function (4) we need to evaluate it in a basis. Using the OAM eigenstates, we get

$$\begin{aligned} W_{\hat{\varrho}}(\ell,\phi) \;&=\; \frac{1}{2\pi}\sum_{\ell'\in\mathbb{Z}} e^{-2i\ell'\phi}\langle\ell-\ell'|\hat{\varrho}|\ell+\ell'\rangle \\ &+\; \frac{1}{2\pi^2}\sum_{\ell',\ell''\in\mathbb{Z}}\frac{(-1)^{\ell''}}{\ell''+1/2}e^{-(2\ell'+1)i\phi} \\ &\times\; \langle\ell+\ell''-\ell'|\hat{\varrho}|\ell+\ell''+\ell'+1\rangle\,. \end{aligned} \tag{9}$$

This looks rather cumbersome due to the second sum in Eq. (9) and sometimes is preferable to work in the angle representation, for which one easily finds

$$W_{\hat{\varrho}}(\ell,\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\langle\phi-\phi'/2|\hat{\varrho}|\phi+\phi'/2\rangle\, e^{i\phi'\ell}\,d\phi'\,. \tag{10}$$

This coincides with the result of Mukunda [31, 32] (see also Ref. [33]) and bears a resemblance with the standard Wigner function for position and momentum that is more than evident. Note that using this latter function in terms of transverse coordinates, as is often done in classical optics [34], is not appropriate for the geometry of the cylinder, which is the natural domain in which the Wigner function should be defined.

We have now all the ingredients needed to accomplish our program. In what follows, the Fourier transform of 2π-periodic functions (i.e., with domain in $\mathcal{S}_1$), defined as

$$(\mathcal{F}g)(k) = \frac{1}{2\pi}\int_{2\pi} g(\phi)\,e^{i\phi k}\,d\phi\,, \tag{11}$$

with $k\in\mathbb{Z}$, will play a relevant role. We first state our main result, which can be viewed as analogous to the Hudson theorem for the canonical pair angle and angular momentum.

Theorem (Classical OAM states)**.** *The Wigner function of a pure state $|\psi\rangle$ is non-negative if and only if $|\psi\rangle$ is an OAM eigenstate $|\ell_0\rangle$.*

Proof. The sufficiency is obvious since the Wigner function for the state $|\ell_0\rangle$ is $W_{|\ell_0\rangle}(\ell,\phi) = \delta_{\ell\ell_0}/(2\pi)$. The delicate point is to prove the necessity. Before proceeding, we sketch the idea behind the proof. The first step is to show that the wave function [and thus, the integrand in Eq. (10)] must be of constant modulus. The second step is then to corroborate that the Wigner function can only be non-zero for a single value of ℓ. Traciality permits us to derive an equation that shows that this value of ℓ cannot vary over ϕ, and that indeed the only states with non-negative Wigner functions are the OAM eigenstates. We start with the following lemma.

Lemma 1. *If the Fourier transform of a smooth, complex, 2π-periodic function $g(\phi)$ is non-negative, then the integration kernel $g(\phi-\phi')$ is non-negative.*

Proof. By a direct calculation we can check that

$$\int_{2\pi} g(\phi-\phi')\, e^{-i\phi' k}\, d\phi' = 2\pi\, (\mathcal{F}g)(k)\, e^{-i\phi k}\,, \tag{12}$$

so, for any smooth test function $\chi(\phi) = \sum_{k\in\mathbb{Z}} \chi(k)\, e^{-i\phi k}$, it holds

$$\int_{2\pi} \chi^*(\phi)\, g(\phi-\phi')\, \chi(\phi')\, d\phi d\phi' = 4\pi^2 \sum_{k\in\mathbb{Z}} |\chi(k)|^2\, (\mathcal{F}g)(k)\,. \tag{13}$$

It is clear that the non-negativity of the kernel $g(\phi-\phi')$ follows from the non-negativity of the Fourier transform $(\mathcal{F}g)(k)$. ☐

We apply the lemma to

$$\chi(\phi) = \frac{1}{2}[\delta_{2\pi}(\phi-c_1) + \delta_{2\pi}(\phi-c_2)]\,, \tag{14}$$

Here, $\delta_{2\pi}$ denotes the periodic delta function (or Dirac comb) of period 2π and $c_1, c_2 \in \mathcal{S}_1$. For this function we have $|\chi(k)|^2 = \{1+\cos[k(c_1-c_2)]\}/(8\pi^2)$, so the sum in the right-hand side of Eq. (13) reduces to

$$g(0)/2 + [g(c_1-c_2) + g(c_2-c_1)]/4\,. \tag{15}$$

Consequently, for a function $g(\phi)$ whose Fourier transform is non-negative, the kernel $g(\phi-\phi')$ must also be non-negative on the test functions (14) for all the possible parameters $c_1, c_2 \in \mathcal{S}_1$.

For a pure state $|\psi\rangle$, the Wigner function (10) is just the Fourier transform of $\psi^*(\phi+\phi'/2)\,\psi(\phi-\phi'/2)$, where we have expressed the wave functions in the angle representation. By Lemma 1, for the test functions (14) the non-negativity of $W_{|\psi\rangle}$ leads to

$$|\psi(\phi)|^2 \geq |\psi(\phi-a/2)|\, |\psi(\phi+a/2)|\,, \tag{16}$$

with $a = c_1 - c_2$. This implies that $|\psi(\phi)|$ cannot have any minima and the modulus of ψ must thus be flat over $\mathcal{S}_1$.

To proceed further we need a technical detail.

Lemma 2. *If a function $f(k) : \mathbb{Z} \to \mathbb{C}$ has an inverse Fourier transform of constant modulus over ϕ, then*

$$\sum_{k\in\mathbb{Z}} f(k)\, f^*(k+j) = 0 \qquad \forall j \neq 0\,. \tag{17}$$

Proof. Let us first introduce the operator

$$\hat{A} = \sum_{m,k\in\mathbb{Z}} f(m-k)\, |m\rangle\langle k|\,. \tag{18}$$

One can check that it can be expressed in a diagonal form in the angle basis, namely

$$\hat{A} = \int_{2\pi} |\phi\rangle\langle\phi|\, (\mathcal{F}^{-1}f)(-\phi)\, d\phi\,. \tag{19}$$

If $(\mathcal{F}^{-1}f)(\phi)$ has constant modulus, it can be written as $(\mathcal{F}^{-1}f)(\phi) = c\, e^{i\lambda(\phi)}$, where λ is a real function. Therefore, we have $\hat{A}\,\hat{A}^\dagger = |c|^2\, \hat{\mathbb{1}}$. But according to the definition (18), this is tantamount to the orthogonality relation

$$\sum_{m,k\in\mathbb{Z}} \sum_{m',k'\in\mathbb{Z}} \langle n|m\rangle\langle k|f(m-k)|k'\rangle\langle m'|f^*(m'-k')|n+j\rangle = 0\,.$$

The Plancherel formula allows one to cancel the diagonal parts, so we are led to

$$\sum_{k\in\mathbb{Z}} f(n-k)\, f^*(n+j-k) = 0\,, \tag{20}$$

whence the result follows. ☐

Next, for every ϕ, we consider the Wigner function of the state as a function exclusively of the discrete index ℓ; that is, $f_\phi(\ell) = W_{|\psi\rangle}(\ell,\phi) : \mathbb{Z} \to \mathbb{R}$ (in fact, W is real valued), and make use of the fact that the (inverse) Fourier transform of $f_\phi(\ell)$ has a constant modulus over ϕ. Then, by Lemma 2, the orthogonality

$$\sum_{\ell\in\mathbb{Z}} f_\phi(\ell)\, f_\phi^*(\ell+\ell') = 0\,, \qquad \forall \ell' \neq 0\,, \tag{21}$$

must hold for all $\phi \in \mathcal{S}_1$. But since f is non-negative on the whole phase-space, this is only possible if f is equal to zero for all but one ℓ_0. Note that, in principle, ℓ_0 may depend on ϕ. Taking into account the marginal distribution (7), we see that $W(\ell,\phi) = \delta_{\ell\ell_0(\phi)}/(2\pi)$.

We now make use of the fact that the state $|\psi\rangle$ is pure [that is, $\mathrm{Tr}(\hat{\varrho}^2) = 1$]. From the traciality property, one can show that the Wigner function representing the product of two density operators $\hat{\varrho}$ and $\hat{\sigma}$ can be expressed as

$$W_{\hat{\varrho}\,\hat{\sigma}}(\ell,\phi) = \frac{1}{2\pi} \sum_{\ell_1\,\ell_2\in\mathbb{Z}} \int_{2\pi} W_{\hat{\varrho}}(\ell+\ell_1, \phi+\psi_1/2) \times W_{\hat{\sigma}}(\ell+\ell_2, \phi+\psi_2/2)\, e^{i(\ell_2\psi_1 - \ell_1\psi_2)}\, d\psi_1 d\psi_2\,. \tag{22}$$

We apply this to the pure state $|\psi\rangle$ whose Wigner function is of the form $\delta_{\ell\ell_0(\phi)}/(2\pi)$.

Without loss of generality, we can assume that $\ell_0(\phi=0) = 0$ and may revert this choice later by a displacement $|\psi\rangle \to \hat{D}(\ell_0, 0)|\psi\rangle$. Then, Eq. (22) becomes

$$W_{|\psi\rangle}(0,0) = \frac{1}{2\pi} = \frac{1}{(2\pi)^3} \int_{2\pi} e^{i[\ell_0(\psi_2/2)\psi_1 - \ell_0(\psi_1/2)\psi_2]}\, d\psi_1 d\psi_2\,. \tag{23}$$

This means that the integral of the imaginary part must vanish, while the integral of the real part must be equal $(2\pi)^2$. This is only possible if the exponential is exactly one for all the arguments (ψ_1, ψ_2); i.e., $\ell_0(\psi_1/2)\,\psi_2 = \ell_0(\psi_2/2)\,\psi_1 \mod 2\pi$. This is only possible when $\ell_0 \equiv 0$. ☐

We have shown that if the Wigner function of a pure state is non-negative, then it is necessarily a Kronecker delta, and thus stems from an OAM eigenstate, which concludes the long yet instructive proof of our theorem.

It is worth stressing that for the continuous case the notions of coherent states, Gaussian wave packets, and states with non-negative Wigner functions (often identified as nonclassical states) are completely equivalent. However, special care must be paid in extending these ideas to other physical systems like OAM, since they lose their equivalence.

For example, OAM coherent states $|\ell_0, \phi_0\rangle$ in the cylinder [35] can be expressed in the angle representation by

$$\langle \phi | \ell_0, \phi_0 \rangle = \frac{e^{i\ell_0(\phi-\phi_0)}}{\sqrt{\vartheta_3\left(0\middle|\frac{1}{e}\right)}} \vartheta_3 \left(\frac{\phi-\phi_0}{2} \middle| \frac{1}{e^2} \right) ,$$

where ϑ_3 denotes the third Jacobi theta function. However, despite the key role played by this function in angular problems, a simple calculation [36] immediately reveals that the Wigner function for them takes negative values.

In the same vein, the states

$$\Psi_\kappa(\phi) = \frac{1}{\sqrt{2\pi I_0(2\kappa)}} \exp(\kappa \cos\phi) , \tag{24}$$

whose associated probability distribution is precisely the von Mises distribution [21], are usually taken as Gaussians for this problem. One can easily check that their Wigner function also takes negative values.

Even with all these cautions, the characterization we have presented of OAM eigenstates as the only ones with non-negative Wigner function has interest in its own, although, unfortunately, they cannot be viewed as Gaussian states.

A topic of interest is the characterization of unitaries that preserve the non-negativity. Obviously, all the displacement operators are of this kind. But the exponential of an arbitrary real function $f(\hat{L})$ also preserves non-negativity and this includes quadratic exponentials, which are essential for a full quantum reconstruction of vortex states [36].

Finally, let us mention that a question that naturally arises is whether our result can be extended to mixed states. Although this question has been approached by using the notion of the Wigner spectrum [37] and explored quite recently for continuous variables [38], in our case a simple extension seems difficult and will be the object of our future work.

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